



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ANNALS OF MATHEMATICS.

VOL. VI.

MARCH, 1892.

No. 5.

ON CERTAIN DETERMINANT FORMS AND THEIR APPLICATIONS.*

(FIRST PAPER.)

By PROF. W. H. ECHOLS, Charlottesville, Va.

Determinants no longer occupy in analysis the position of mere symbols of tabulated results. In the higher analysis they enter largely into the operations themselves. Not only are they powerful levers in these far-reaching methods of investigation, but are as well the instruments for discovering new relations which may exist between quantities and functions when bound together by their side lines. While there may be little or nothing that is new in the results here obtained, the operations serve somewhat to illustrate in an elementary manner the sweeping comprehensiveness of determinant forms.

Consider the alternant determinant function,

$$Fx \equiv \begin{vmatrix} fx, & 1, & x, & x^2, & \dots, & x^n \\ fa_n, & 1, & a_n, & a_n^2, & \dots, & a_n^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ fa_0, & 1, & a_0, & a_0^2, & \dots, & a_0^n \end{vmatrix}. \quad (1)$$

Let the function fx and its first n derived functions be finite and continuous for all values of the argument x from a_0 to a_n .

Let

$$\begin{aligned} a_0 &< a_1 < a_2 < a_3 \dots < a_n, \\ b_0 &< b_1 < b_2 \dots < b_{n-1}, \\ c_0 &< c_1 \dots < c_{n-2}, \\ &\dots \dots \dots \dots \dots \dots \dots \\ w_0 &< w_1, \\ &u \end{aligned}$$

* A paper read before the Mathematical Section of the University of Virginia Philosophical Society.

be quantities such that

$$\begin{aligned} a_r &< b_r < a_{r+1}, \\ b_r &< c_r < b_{r+1}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ w_0 &< u < w_1. \end{aligned}$$

I. GENERALIZATION OF ROLLE'S THEOREM.

The modern form of Rolle's theorem is this : " If a continuous function fx vanish when $x = a$, and also when $x = b$, then its derived function $f'x$, if also continuous, must vanish for some value of x between a and b ." The proof of it goes hand in hand with the conception of a derived function.

Lagrange's form of Rolle's theorem is this : " If fx be a continuous function for all values of x between $x = a$ and $x = b$, then

$$fu = \frac{fa - fb}{a - b},$$

where u is some value of x between a and b ."

It is proposed to generalize this form of the theorem, and to express f^nu in terms of $fa_0, \dots, fu_n, a_0, \dots, a_n$.

The determinant function Fx vanishes whenever x takes any one of the $n + 1$ values a_0, \dots, a_n . Its first derived function, therefore, vanishes whenever x takes any one of the n values b_0, \dots, b_{n-1} . So on, until finally its n th derived function vanishes for some value u of x , such that $a_0 < u < a_n$.

The n th derived function of Fx is, as may easily be seen,

$$F^nx = Mf^nx + (-1)^{n+1}n!M_n,$$

where M is the minor of fx , and M_n that of x^n in the determinant Fx .

Hence

$$\begin{aligned} f^nu &= (-1)^n n! \frac{M_n}{M}, \\ &= n! \frac{\begin{vmatrix} 1, & a_n, & \dots, & a_n^{n-1}, & fa_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & a_0, & \dots, & a_0^{n-1}, & fa_0 \end{vmatrix}}{\begin{vmatrix} 1, & a_n, & \dots, & a_n^{n-1}, & a_n^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & a_0, & \dots, & a_0^{n-1}, & a_0^n \end{vmatrix}}, \end{aligned}$$

which is the desired relation.

If we represent the minors of fa_n, \dots, fa_0 in the determinant M_n by A_n, \dots, A_0 , respectively, we have

$$f^nu = \frac{n!}{M} (A_0 fa_0 + A_1 fa_1 + \dots + A_n fa_n), \quad (2)$$

where M and the A 's contain only the a 's, and are therefore independent of the form of the function.

COROLLARY I.—If $fa_0 = fa_1 = \dots = fa_n = fa$, then

$$Fx = M(fx \pm fa),$$

and

$$F^rx = Mf^rx;$$

whence the derivatives of Fx and fx vanish together.

Since

$$Fx = 0, \quad \text{for } x = a_0, \dots, a_n,$$

we have

$$F'b_0 = 0, F'b_1 = 0, \dots, F'b_{n-1} = 0,$$

$$\therefore f'b_0 = 0, f'b_1 = 0, \dots, f'b_{n-1} = 0.$$

In like manner,

$$f''c_0 = 0, f''c_1 = 0, \dots,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f^nu = 0.$$

The same results hold in the particular case when

$$fa_0 = 0, fa_1 = 0, \dots, fa_n = 0,$$

which is the extended form of Rolle's theorem due to M. Ossian Bonnet.

COROLLARY II.—If all of the different values of fa , except fa_r, fa_s, fa_t, \dots , are equal to each other and to fa , we have

$$\begin{aligned} M \frac{f^nu}{n!} &= (A_0 + A_1 + \dots + A_n) fa - (A_r + A_s + A_t) fa \\ &\quad + A_r fa_r + A_s fa_s + A_t fa_t \\ &= A_r (fa_r - fa) + A_s (fa_s - fa) + A_t (fa_t - fa), \end{aligned}$$

since $(A_0 + A_1 + \dots + A_n)$ is equal to the determinant formed by replacing each of the fa 's by unity, and therefore vanishes.

If all are equal except one, fa_r , then, writing out the binomial difference-products of the alternants,*

$$\begin{aligned} fa_r - fa &= \frac{M}{A_r} \frac{f^nu}{n!}, \\ &= (-1)^r (a_r - a_0) (a_r - a_1) \dots (a_r - a_n) \frac{f^nu}{n!}. \end{aligned}$$

* Muir's Determinants, p. 162, *et seq.*

Corresponding results hold when the fa values vanish as a particular case. Thus, putting x for a_r , and

$$fa_0 = 0, \quad fa_1 = 0, \quad \dots, \quad fa_n = 0,$$

we have

$$fx = (x - a_0)(x - a_1) \dots (x - a_n) \frac{f^{nu}}{n!},$$

a form also due to M. Ossian Bonnet.

When $n = 1$, we have the following geometrical illustration of Lagrange's form of Rolle's theorem:

$$Fx = \begin{vmatrix} fx, & x, & 1 \\ fa_1, & a_1, & 1 \\ fa_0, & a_0, & 1 \end{vmatrix}$$

is the double area of the triangle whose base is the chord $(fa_1, a_1), (fa_0, a_0)$ of the curve fx , the vertex of the triangle being a running point on the curve.

This area vanishes when the vertex coincides with either extremity of the chord. So that, if the function fx is finite, continuous, and single-valued between $x = a_1$ and $x = a_0$, we have the first derivative of Fx vanishing for some value u , of x , between a_0 and a_1 . Hence,

$$fa_1 - fa_0 = (a_1 - a_0) f'u.$$

The chord being regarded as of constant length, the change of Fx is therefore proportional to the altitude of the triangle. As the length of the chord converges to zero, the altitude of the triangle for $x = u$ converges to the mid-ordinate of the segment of the osculating circle on that chord.

II. INTERPOLATION FORMULÆ.

If all of the $n + 1$ values of fa are known except fa_r , we have by (2)

$$fa_r = - \left[\frac{A_0}{A_r} fa_0 + \frac{A_1}{A_r} fa_1 + \dots + \frac{A_n}{A_r} fa_n \right] + \frac{M}{A_r} \frac{f^{nu}}{n!}.$$

Writing out the alternants into binomial difference-products and cancelling common factors in numerator and denominator of the coefficients, we have

$$\begin{aligned} (-1)^r fa_r &= \frac{(a_r - a_1)(a_r - a_2) \dots (a_r - a_n)}{(a_0 - a_1)(a_0 - a_2) \dots (a_0 - a_n)} fa_0 + \frac{(a_r - a_0)(a_r - a_2) \dots (a_r - a_n)}{(a_1 - a_0)(a_1 - a_2) \dots (a_1 - a_n)} fa_1 \\ &\quad \dots + \frac{(a_r - a_0)(a_r - a_1) \dots (a_r - a_{n-1})}{(a_n - a_0)(a_n - a_1) \dots (a_n - a_{n-1})} fa_n \\ &\quad + (-1)^r (a_r - a_0)(a_r - a_1) \dots (a_r - a_n) \frac{f^{nu}}{n!}. \end{aligned}$$

If we leave off the last term, which renders the equation exact, we have Lagrange's interpolation formula, the most important one in analysis, the principle of which is to substitute for the unknown function a rational integral function of the $(n-1)$ th degree which has in common with it the n values fa_0, \dots, fa_n . The residual term in f^nu corresponds to the remainder after n terms in Taylor's formula, and shows that if the unknown function be a rational integral function of no higher degree than $n-1$, Lagrange's formula is exact, since $f^nu = 0$. If the unknown function be of the n th degree and the coefficient of x^n be unity, then the formula above gives exact results, since $f^nu = n!$.

The above result was written down merely to identify (1) with Lagrange's formula. The interpolation formula is written out at once from the original determinant; thus, letting M_0, M_1, \dots, M_n be the minors of $1, x, \dots, x^n$ in Fx , we have

$$\begin{aligned} fx &= - \left[\frac{M_0}{M} + \frac{M_1}{M} x + \frac{M_2}{M} x^2 + \dots + \frac{M_n}{M} x^n \right] + \frac{Fx}{M} \\ &= \varphi x + \frac{Fx}{M}. \end{aligned} \quad (3)$$

In this, Fx vanishing for $x = a_0, \dots, a_n$, makes fx and the rational integral function φx , of the n th degree, have $n+1$ common values. Neglecting Fx/M , we interpolate any function fx for a given x between a_0 and a_n . Computing the coefficients in φx once for all, any number of such interpolations may be rapidly made.

Fx/M being a function which vanishes for the $n+1$ values $x = a_0, \dots, a_n$, we may write it

$$(x - a_0)(x - a_1) \dots (x - a_n) \psi x.$$

The form of ψx^* depending in general on that of fx .

If fx be a rational integral function of the n th degree, so also must Fx be. But Fx vanishes $n+1$ times; it is therefore zero for all values of x , and the interpolation is exact. If fx be of the $(n+1)$ th degree, so is Fx . Hence ψx cannot contain x , and is constant.

If in (2) we put the factors in the coefficients of fa_i , etc. equal, so that

$$a_0 - a_1 = a_1 - a_2 = \dots = \Delta a = (a_0 - a_n)/n = h/n;$$

we have

$$\frac{h^n}{n^n} \frac{f^nu}{n!} = \frac{fa_0}{n!} - \frac{fa_1}{(n-1)!} + \dots + (-1)^r \frac{fa_r}{r!(n-r)!} \dots + (-1)^n \frac{fa_n}{n!}.$$

* It is shown below that $\psi x = (-1)^{n+1} \frac{f^{n+1}v}{(n+1)!}$, where $a_0 < v < a_n$.

If $h/n = 1$, then

$$f^n u = f a_0 - n f a_1 + \frac{n(n-1)}{2!} f a_2 - \dots + (-1)^n f a_n,$$

a formula analogous to the fundamental interpolation formula of finite differences.

In (3) the coefficient M_p/M of x^p is the same as that of x^n , if in M_n we change a_p into a_n , and multiply by $(-1)^{n-p}$.

III. MECHANICAL QUADRATURE.

The area of the curve fx is, by (3),

$$\int_{a_0}^{a_n} fx \, dx = \int_{a_0}^{a_n} \varphi x \, dx + \int_{a_0}^{a_n} \frac{Fx}{M} \, dx.$$

Approximately

$$\int_{a_0}^{a_n} fx \, dx = \int_{a_0}^{a_n} \varphi x \, dx,$$

the error being

$$\int_{a_0}^{a_n} \frac{Fx}{M} \, dx.$$

The curve Fx/M cuts the axis $n+1$ times at the points $x = a_0, \dots, a_n$, and the area is made up of positive and negative portions which tend to annul each other. It seems probable that with a proper distribution of the $n-2$ intermediate points a_1, \dots, a_{n-1} , and frequently of but one or more of them, the error area may be made to vanish.

Gauss (*Werke*, Vol. III, p. 203) shows how to make this distribution independently of the form of fx , when fx is a rational integral function of a degree not higher than $2n$ (see Jacobi's proof; Boole's *Finite Differences*, p. 52). Gauss's quadrature formula is the integration of Lagrange's interpolation formula, in which the error area is

$$\int_{a_0}^{a_n} (-1)^r (a_r - a_0) \dots (a_r - a_n) \frac{f^n u}{n!} da_r.$$

As a simple illustration of the error vanishing when $f x$ is an irrational function, the approximate formula gives exactly the quadrant of a circle (radius = 1) for $a_0 = 0$, $a_2 = 1$, and a_1 having some value between $\frac{37}{60}$ and $\frac{375}{600}$, the error for these values being $+0.0006$ and -0.00165 , respectively.

Of course, if the ordinates be equidistant, the general formula reduces to the ordinary well known forms.

The most important researches in the theory of interpolation have had reference to this formula. The investigations of Minding, Christoffel, Mehler, and Jacobi (all published in Crelle's Journal; see Boole, Finite Diff., p. 56) have had in view the general problem of determining the position of the ordinates we should choose, so that the approximation may be as close as possible. There still remains much to be done in this direction. The great importance of the computation of definite integrals is sufficient cause for close research in the attempt to reduce the error area for the general definite integral.

We may examine

$$\int_{a_0}^{a_n} \frac{F x}{M} dx$$

in the following manner:—

The original determinant suggests that we may put

$$F x \equiv \begin{vmatrix} 1, & x, & x^2, & \dots, & x^{n+1} \\ 1, & a_n, & a_n^2, & \dots, & a_n^{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & a_0, & a_0^2, & \dots, & a_0^{n+1} \end{vmatrix} R \equiv \Delta R,$$

where R is some unknown function of x . When x has some fixed definite value x_0 , we have

$$R_0 = \frac{F x_0}{\Delta_0},$$

which is independent of x .

The function

$$F x - \Delta R_0$$

vanishes when x has any one of the $n + 2$ values a_0, \dots, a_n, x_0 . Its $(n + 1)$ th derivative

$$F^{n+1} x - R_0 \frac{d^{n+1}}{dx^{n+1}} \Delta,$$

or

$$M f^{n+1} x - (-1)^{n+1} (n + 1)! M R_0,$$

will therefore vanish for some value of x , say v , such that v lies between α_n and α_0 , the greatest and least of these values.* Hence

$$R_0 = (-1)^{n+1} \frac{f^{n+1}v}{(n+1)!}.$$

Since x_0 may be any value of x between α_0 and α_n , we may drop the suffix, and put

$$R = (-1)^{n+1} \frac{f^{n+1}v}{(n+1)!}.$$

* It is easy to show that Prof. Sylvester's general alternant

$$\begin{vmatrix} f_1x, & f_2x, & \dots, & f_{n+1}x \\ f_1x_1, & f_2x_1, & \dots, & f_{n+1}x_1 \\ \dots & \dots & \dots & \dots \\ f_1x_n, & f_2x_n, & \dots, & f_{n+1}x_n \end{vmatrix}$$

may be treated in the same manner, obtaining the relation

$$\begin{vmatrix} f_1x, & f_2x, & \dots, & f_rx, & \dots, & f_sx, & \dots, & f_{n+2}x \\ f_1x_1, & f_2x_1, & \dots, & f_rx_1, & \dots, & f_sx_1, & \dots, & f_{n+2}x_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ f_1x_n, & f_2x_n, & \dots, & f_rx_n, & \dots, & f_sx_n, & \dots, & f_{n+2}x_n \\ 0, & 0, & \dots, & \varphi^{n+1}u, & \dots, & 1, & \dots, & 0 \end{vmatrix} = 0,$$

in which the last constituent of each column except the r th and the s th is zero, and

$$\varphi^{n+1}u = \frac{F_u^{n+1}}{\Delta_u^{n+1}},$$

where

$$F_u^{n+1} = \begin{vmatrix} f_1^{n+1}u, & f_2^{n+1}u, & \dots, & f_s^{n+1}u, & \dots, & f_{n+2}^{n+1}u \\ f_1x_1, & f_2x_1, & \dots, & f_sx_1, & \dots, & f_{n+2}x_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ f_1x_n, & f_2x_n, & \dots, & f_sx_n, & \dots, & f_{n+2}x_n \end{vmatrix},$$

in which the r th column is wanting, and

$$\Delta_u^{n+1} = \begin{vmatrix} f_1^{n+1}u, & f_2^{n+1}u, & \dots, & f_r^{n+1}u, & \dots, & f_{n+2}^{n+1}u \\ f_1x_1, & f_2x_1, & \dots, & f_rx_1, & \dots, & f_{n+2}x_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ f_1x_n, & f_2x_n, & \dots, & f_rx_n, & \dots, & f_{n+2}x_n \end{vmatrix},$$

in which the s th column is wanting.

The quantity u has some value between the greatest and least of the values x_1, \dots, x_n , and is to be substituted for x after the $(n+1)$ th differentiation.

Therefore

$$\int_{a_0}^{a_n} \frac{Fx}{M} dx = \int_{a_0}^{a_n} (-1)^{n+1} \frac{A}{M} \frac{f^{n+1}v}{(n+1)!},$$

the same result as before.*

Any effort to reduce this, when the form of the function is known, must be made through an inquiry into the value and form of v ,† which is a function of $f(x)$, a_0 , a_n , and x .

GENERALIZATION OF THE EXPANSION OF fx .

We begin by deducing Taylor's formula in determinant notation and then proceed to the generalization of the expansion of the function fx .

1. *Taylor's Formula*.—It is required to find a relation, if such exists, between the two values of a function, fx and $f(x+h)$, the first n integral powers of h , and the first n derivatives of fx .

Write down in a line, $f(x+h)$ followed by the successive powers of h up to the n th inclusive. Underneath this line write a second line formed by putting $h=0$ in the first line. Underneath this second line write a third line formed by differentiating the first line with respect to h , and in the result putting $h=0$. Repeat this operation until the first line has been differentiated n times. Draw the side lines, thus forming a determinant function D . This determinant D is a function of the quantities whose inter-relation is desired.

$$D = \begin{vmatrix} f(x+h), & 1, & h, & h^2, & h^3, & \dots, & h^{n-1}, & h^n \\ f_x, & 1, & 0, & 0, & 0, & \dots, & 0, & 0 \\ f'_x, & 0, & 1!, & 0, & 0, & \dots, & 0, & 0 \\ f''_x, & 0, & 0, & 2!, & 0, & \dots, & 0, & 0 \\ f'''_x, & 0, & 0, & 0, & 3!, & \dots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1}x, & 0, & 0, & 0, & 0, & \dots, & (n_1-1)!, & 0 \\ f^n x, & 0, & 0, & 0, & 0, & \dots, & 0, & n! \end{vmatrix}$$

We observe that D and each of its derivatives down to the n th, inclusive, vanishes if h vanishes.

* See Synopsis der Hoheren Mathematik von Johann G. Hagen, Erster Band, pp. 158, 207, or Baltzer, Theorie der Determinanten, p. 87, where the alternant Fx may be found, and notes on Gauss's quadrature.

† See a paper on the remainder after n terms in Taylor's formula, by A. W. Whitcom, American Journal of Math., Vol. III, No. 4, which indirectly bears on this subject.

Factoring out $1!, 2!, \dots, n!$, from the 2nd, 3rd, \dots , $(n+1)$ th columns, respectively, we have*

$$n!! \begin{vmatrix} f(x+h), 1, \frac{1}{1!}h, \frac{1}{2!}h^2, \dots, \frac{1}{n!}h^n \\ fx, & 1, & 0, & 0, & \dots, & 0 \\ f'x & 0, & 1, & 0, & \dots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^nx, & 0, & 0, & 0, & \dots, & 1 \end{vmatrix} - D \equiv 0.$$

Observing that in the determinant the general term is

$$(-1)^{2r+1} \frac{h}{r!} f^rx,$$

we have

$$f(x+h) = fx + hf'x + \dots + \frac{h^n}{n!} f^nx + \frac{D}{n!!}.$$

This series is convergent if D is finite, since by increasing n sufficiently we may make $D/n!!$ as small as we choose.

The form of the determinant suggests putting

$$D = n!! \frac{h^{n+1}}{(n+1)!} R,$$

where R is some *unknown* function of x and h .

When h has some definite fixed value h_0 , we have

$$D_0 = n!! \frac{h_0^{n+1}}{(n+1)!} R_0,$$

or

$$R_0 = \frac{(n+1)!}{n!! h_0^{n+1}} D_0,$$

which is independent of h .

Then the function

$$\begin{vmatrix} f(x+h), 1, \frac{1}{1!}h, \dots, \frac{1}{n!}h^n, \frac{1}{(n+1)!}h^{n+1} \\ fx, & 1, & 0, & \dots, & 0, & 0 \\ f'x, & 0, & 1, & \dots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^nx & 0, & 0, & \dots, & 1, & 0 \\ R_0, & 0, & 0, & \dots, & 0, & 1 \end{vmatrix}$$

* I use the symbol $n!!$ to represent the double factorial product $1! 2! 3! \dots n!$

and each of its derivatives to the n th, inclusive, vanish when $h = 0$. But the function vanishes when $h = h_0$. Its first derivative vanishes therefore for some value h_1 between 0 and h_0 . Its second derivative vanishes also for h_2 , such that $0 < h_2 < h_1$; and so on, until finally its $(n + 1)$ th derivative vanishes for some value of h , say θh , such that $0 < \theta h < h_n < h_0$.

The $(n + 1)$ th derivative of the above determinant is

$$f^{n+1}(x + h) - R_0;$$

hence

$$R_0 = f^{n+1}(x + \theta h).$$

We have, therefore,

$$D_0 = n!! \frac{h_0^{n+1}}{(n + 1)!} f^{n+1}(x + \theta h).$$

Dropping the suffix, because h_0 is any finite value of h such that fx and its first $(n + 1)$ derivatives are finite and continuous between the limits x and $x + h$, we have, finally,

$$\begin{vmatrix} f(x + h), & 1, & h, & \dots, & \frac{1}{(n + 1)!} h^{n+1} \\ fx, & 1, & 0, & \dots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x, & 0, & 0, & \dots, & 1, & 0 \\ f^{n+1}(x + \theta h), & 0, & 0, & \dots, & 0, & 1 \end{vmatrix} \equiv 0. \quad (4)$$

The original determinant was filled in with zeros merely in order to simplify its expansion. It might, however, have been written in the more general form

$$\begin{vmatrix} f(x + h), & 1, & (x + h), & (x + h)^2, & \dots, & (x + h)^n \\ fx, & 1, & x, & x^2, & \dots, & x^n \\ f'x, & 0, & 1, & 2x, & \dots, & nx^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x, & 0, & 0, & 0, & \dots, & n! \end{vmatrix},$$

and the result in corresponding shape.

2. *Generalization of the Expansion of fx .*—While, in substance, the preceding method is but Maclaurin's theorem, yet it may be regarded as an extension of it, inasmuch as it affords quite an independent method of expanding any function in powers of a variable. For while, after Maclaurin's method, we have successively after differentiation put the variable equal to zero, we might have put it equal to any constant, in order to make the successive derivatives

independent of the variable. Or, indeed, we may substitute for the variable, after differentiation, any series of constants,* as in the following :—

Let φx be any function of x , finite and continuous, together with its first n derivatives, for the values a_0, \dots, a_n of the variable.

Write down φx followed by the n powers of x . Underneath write these values when $x = a_0$. Underneath write the derivative of the first line, and substitute in it a_1 for x , and so on until the n th derivative is written with a_n for x . Draw the side lines and call the determinant F .

Factor out the factorials in the diagonal as before, giving the result ; thus,

$$n!! \begin{vmatrix} \varphi x, & 1, & x, & \frac{1}{2!}x^2, & \dots, & \frac{1}{n!}x^n \\ \varphi a_0, & 1, & a_0, & \frac{1}{2!}a_0^2, & \dots, & \frac{1}{n!}a_0^n \\ \varphi' a_1, & 0, & 1, & a_1, & \dots, & \frac{1}{(n-1)!}a_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^n a_n, & 0, & 0, & 0, & \dots, & 1 \end{vmatrix} \equiv F.$$

Let

$$F \equiv (-1)^{n+1} n!! \begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \dots, & \frac{1}{n!}x^n, & \frac{1}{(n+1)!}x^{n+1} \\ 1, & a_0, & \frac{1}{2!}a_0^2, & \dots, & \frac{1}{n!}a_0^n, & \frac{1}{(n+1)!}a_0^{n+1} \\ 0, & 1, & a_1, & \dots, & \frac{1}{n!}a_1^n & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \dots, & 1, & a_n \end{vmatrix} R,$$

$$\equiv (-1)^{n+1} n!! \Delta R.$$

Where R is some *unknown* function of x .

When x has some fixed finite value x_0 , we have

$$R_0 = (-1)^{n+1} \frac{F_0}{n!! \Delta_0},$$

which is independent of x .

The function

$$F - (-1)^{n+1} n!! \Delta R_0,$$

or

$$\begin{vmatrix} \varphi x, & 1, & x, & \dots, & \frac{1}{(n+1)!}x^{n+1} \\ \varphi a_0, & 1, & a_0, & \dots, & \frac{1}{(n+1)!}a_0^{n+1} \\ \varphi' a_1, & 0, & 1, & \dots, & \frac{1}{n!}a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^n a_n, & 0, & 0, & \dots, & 1, & a_n \\ R_0, & 0, & 0, & \dots, & 0, & 1 \end{vmatrix}$$

* Throughout this paper I mean by constants, in the sense as employed above, any quantities, or in general any functions, which are independent of x .

vanishes when $x = a_0$, also when $x = x_0$; therefore its first derivative vanishes for some value of x , say x_1 , between a_0 and x_0 . Since its first derivative vanishes for $x = x_1$, and also for $x = a_1$, its second derivative must vanish for some value of x , say x_2 , between x_1 and a_1 . And so on, until finally its $(n + 1)$ th derivative vanishes for some value of x , say u , between x_n and a_n .

$$\therefore R_0 = \varphi^{n+1}u.$$

Hence

$$F_0 - (-1)^{n+1} n!! \Delta_0 \varphi^{n+1}u \equiv 0.$$

In this equation we may drop the suffix, since x_0 is but *any* fixed finite value which x may have, within the limits of continuity. We have, then, in general,

$$F - (-1)^{n+1} n!! \Delta \varphi^{n+1}u \equiv 0,$$

or

$$\begin{vmatrix} \varphi x, & 1, & x, & \dots, & \frac{1}{(n+1)!} x^{n+1} \\ \varphi a_0, & 1, & a_0, & \dots, & \frac{1}{(n+1)!} a_0^{n+1} \\ \varphi' a_1, & 0, & 1, & \dots, & \frac{1}{n!} a_1^n \\ \dots & \dots & \dots & \dots & \dots \\ \varphi^n a_n, & 0, & 0, & \dots, & 1, & a_n \\ \varphi^{n+1}u, & 0, & 0, & \dots, & 0, & 1 \end{vmatrix} \equiv 0. \quad (5)$$

We bring $\varphi^r a_r$ to the first place in the first row by $r + 1$ interchanges; so we have for the corresponding term in the expansion of the determinant

$$(-1)^{r+1} A_r \varphi^r a_r;$$

and when transposed it becomes

$$(-r) A_r \varphi^r a_r.$$

Whence the series

$$\varphi x = \varphi a_0 - A_1 \varphi' a_1 + A_2 \varphi'' a_2 - \dots + (-1)^n A_n \varphi^n a_n + (-1)^{n+1} A_{n+1} \varphi^{n+1}u, \quad (6)$$

in which the coefficients A_1, \dots, A_{n+1} , are the minors of $\varphi a_1, \dots, \varphi^n a_n$ in the above determinant, and are independent of the form of the function.

If all of the values a_1, a_2, \dots, a_n become equal to any one of them, say a_0 , or if the values a_1, a_2, \dots, a_n become equal to x , the series becomes Taylor's expansion; and if $a_0 = a_1 = \dots a_n = 0$, it becomes Maclaurin's. If $a_0 = 0$, and $a_1 = a_2 = \dots = a_n = x$, we have John Bernoulli's series.

In order to avoid the tedium of expanding the determinant coefficients A_1, \dots, A_n , we observe the following:—

$$\begin{aligned}
 A_r &= \begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \dots, & \frac{1}{r!}x^r \\ 1, & a_0, & \frac{1}{2!}a_0^2, & \dots, & \frac{1}{r!}a_0^r \\ 0, & 1, & a_1, & \dots, & \frac{1}{(r-1)!}a_1^{r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \dots, & 1, & a_{r-1} \end{vmatrix} \\
 &= - \left\{ \begin{vmatrix} x, & \frac{1}{2!}x^2, & \dots, & \frac{1}{r!}x^r \\ 1, & a_1, & \dots, & \frac{1}{(r-1)!}a_1^{r-1} \\ 0, & 1, & \dots, & \frac{1}{(r-2)!}a_2^{r-2} \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \dots, & 1, & a_{r-1} \end{vmatrix} - \begin{vmatrix} a_0, & \frac{1}{2!}a_0^2, & \dots, & \frac{1}{r!}a_0^r \\ 1, & a_1, & \dots, & \frac{1}{(r-1)!}a_1^{r-1} \\ 0, & 1, & \dots, & \frac{1}{(r-2)!}a_1^{r-2} \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \dots, & 1, & a_{r-1} \end{vmatrix} \right\} \\
 &= - \int_{a_0}^x \begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \dots, & \frac{1}{(r-1)!}x^{r-1} \\ 1, & a_1, & \frac{1}{2!}a_1^2, & \dots, & \frac{1}{(r-2)!}a_1^{r-2} \\ 0, & 1, & a_2, & \dots, & \frac{1}{(r-3)!}a_2^{r-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \dots, & 1, & a_{r-1} \end{vmatrix} dx.
 \end{aligned}$$

But

$$A_{r-1} = \begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \dots, & \frac{1}{(r-1)!}x^{r-1} \\ 1, & a_0, & \frac{1}{2!}a_0^2, & \dots, & \frac{1}{(r-1)!}a_0^{r-1} \\ 0, & 1, & a_1, & \dots, & \frac{1}{(r-2)!}a_1^{r-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \dots, & 1, & a_{r-2} \end{vmatrix}$$

Therefore, we observe that any A may be derived from the preceding one by an operation analogous to that of integration. Integrate the variable according to the regular rules for integration, and at the same time increase all the suffixes of the arbitrary constants by one. Thus, using the symbol I to denote the operation,* we have

$$A_r = - \int_{a_0}^x A_{r-1} dx.$$

* The symbol I and that of the inverse operator Q should in addition include the change of algebraic sign.

The functions $\varphi^r a_r$, independent of x , may also be included in the operation by increasing both the suffix of the argument and the index of the function by unity. Thus

$$\begin{aligned} A_r \varphi^r a_r &= - \int_{a_0}^x A_{r-1} \varphi^{r-1} a_{r-1} dx. \\ &= (-1)^r \int_{a_0}^x \varphi^0 a_0 dx. \end{aligned}$$

We write the series, finally,

$$\varphi x = \varphi a_0 + \sum_{i=0}^{i=n} \int_{a_0}^x A_i \varphi^i a_i dx + \varphi^{n+1} u \int_{a_0}^x A_1 dx, \quad (7)$$

in which the symbol $\int dx$ means that the above defined operation is to be repeated n times.

A_r is a homogeneous function of the $r+1$ quantities x_1, a_0, \dots, a_{r-1} . In general A_r may be made to vanish by giving either a_{r-1} , or a_{r-2}, \dots, a_1 any one of the one, or two, $\dots, (r-2)$ values which satisfy $A_r = 0$. For example,

$$A_2 = 0, \text{ when } a_1 = \frac{1}{2}(x + a_0).$$

To give the series arithmetical meaning, let x have some fixed definite value $a = a_0 + h$; let a_1, \dots, a_n be intermediate values in the ascending order of magnitude between the fixed limits a_0 and a , between which limits φx and its first n derivatives are finite and continuous for all values of x .

In A_r put first

$$a_1 = a_2 = \dots = a_{r-1} = a_0,$$

the inferior limit. Then A_r becomes

$$(-1)^r \frac{(a - a_0)^r}{r!}.$$

Now put

$$a_1 = a_2 = \dots = a_{r-1} = a,$$

the superior limit. Then A_r becomes

$$(-1)^{r+1} \frac{(a - a_0)^r}{r!}.$$

By taking r sufficiently large as we may make these two values differ as little from zero as we choose.

In general the value of any co-efficient A_r will depend upon the distribution of the intermediate values a_1, \dots, a_{r-1} , and may be made as small as we

please, or be made to vanish, by letting any one of the a 's have a particular value. We have, for computing the A 's, the formula

$$A_r = a_{r-1} A_{r-1} - \frac{1}{2!} a_{r-2}^2 A_{r-2} + \dots + (-1)^{n+1} \frac{a_0^r}{r!} + (-1)^r \frac{x^r}{r!}. \quad (8)$$

For example, from this we obtain

$$A_1 = -(x - a_0),$$

$$A_2 = +\frac{1}{2!} (x^2 - a_0^2) - a_1 (x - a_0),$$

$$A_3 = -\frac{1}{3!} (x^3 - a_0^3) + \frac{1}{2!} a_2 (x^2 - a_0^2) - a_1 a_2 (x - a_0) + \frac{1}{2!} a_1^2 (x - a_0),$$

and so on.

To investigate the relation which must hold between the a 's in order that the series may be convergent, after some fixed term, we have by the above formula

$$\begin{aligned} \frac{A_r}{A_{r-1}} &= a_{r-1} - \frac{1}{2} \frac{a_{r-2}}{A_{r-1}} \left\{ a_{r-2} A_{r-2} - \frac{1}{3} \frac{a_{r-3}^3}{a_{r-2}} A_{r-3} + \frac{1}{3.4} \frac{a_{r-4}^4}{a_{r-2}} A_{r-4} - \dots \right\} \\ &= a_{r-1} - \frac{1}{2} a_{r-2} \frac{a_{r-2} A_{r-2} - \frac{1}{3} \frac{a_{r-3}^3}{a_{r-2}} A_{r-3} + \frac{1}{3.4} \frac{a_{r-4}^4}{a_{r-2}} A_{r-4} - \dots}{a_{r-2} A_{r-2} - \frac{1}{2!} \frac{a_{r-3}^2}{a_{r-3}} A_{r-3} + \frac{1}{3!} \frac{a_{r-4}^3}{a_{r-4}} A_{r-4} - \dots}. \end{aligned}$$

In the ratio of the second term of the member on the right the first terms of the numerator and denominator are identical, while each succeeding term of the numerator is less than the corresponding term of the denominator, since generally $a_r > a_{r-1}$, the ratio in question is therefore less than unity.

Letting $A_r/A_{r-1} = \rho$, we have, therefore,

$$a_{r-1} > \rho > a_{r-1} - \frac{1}{2} a_{r-2};$$

also, a priori,

$$a_{r-1} > \rho > \frac{1}{2} a_{r-1}.$$

Therefore, if a_n lies between the limits 0 and +1, or 0 and -1, the series is convergent.

If we arrange the series according to the powers of x , we have

$$\varphi x = C_0 - C_1 x + C_2 \frac{x_2}{2!} - \dots + (-1)^{n+1} C_{n+1} \frac{x^{n+1}}{(n+1)!}, \quad (9)$$

wherein the determinants C_0, \dots, C_{n-1} , are independent of x . We have

$$\begin{aligned}
 C_0 &= \begin{vmatrix} \varphi a_0, & a_0, & \frac{1}{2!} a_0^2, & \dots, & \frac{1}{(n+1)!} a_0^{n+1} \\ \varphi' a_1, & 1, & a_1, & \dots, & \frac{1}{n!} a_1^n \\ \varphi'' a_2, & 0, & 1, & \dots, & \frac{1}{(n-1)!} a_2^{n-1} \\ . & . & . & . & . \\ \varphi^n a_n, & 0, & 0, & \dots, & 1, & a_n \\ \varphi^{n+1} u, & 0, & 0, & \dots, & 0, & 1 \end{vmatrix} \\
 &= \varphi a_0 - a_0 \varphi' a_1 + \varphi'' a_2 \begin{vmatrix} a_0, & \frac{1}{2!} a_0^2 \\ 1, & a_1 \end{vmatrix} - \varphi''' a_3 \begin{vmatrix} a_0, & \frac{1}{2!} a_0^2, & \frac{1}{3!} a_0^3 \\ 1, & a_1, & \frac{1}{2!} a_1^2 \\ 0, & 1, & a_2 \end{vmatrix} + \dots + \text{etc} \dots \\
 &= \varphi a_0 - \beta_1 \varphi' a_1 + \beta_2 \varphi'' a_2 - \dots (-)^{n+1} \beta_{n+1} \varphi^{n+1} u,
 \end{aligned}$$

with the relation

$$\beta_r = a_{r-1} \beta_{r-1} - \frac{1}{2!} a_{r-2}^2 \beta_{r-2} + \dots (-)^{r+1} \frac{a_0^r}{r!}.$$

As before,

$$\frac{\beta_r}{\beta_{r-1}} = a_{r-1} - \frac{1}{2} a_{r-2} + \epsilon, \quad 0 > \epsilon > -1;$$

and

$$a_{r-1} > \beta_r / \beta_{r-1} > \frac{1}{2} a_{r-1}.$$

Again, we have

$$\begin{aligned}
 C_1 &= -\varphi' a_1 + \varphi'' a_2 \begin{vmatrix} 1, & \frac{1}{2!} a_0^2 \\ 0, & a_1 \end{vmatrix} - \varphi''' a_3 \begin{vmatrix} 1, & \frac{1}{2!} a_0^2, & \frac{1}{3!} a_0^3 \\ 0, & a_1, & \frac{1}{2!} a_1^2 \\ 0, & 1, & a_2 \end{vmatrix} + \dots
 \end{aligned}$$

$$\therefore C_1 x = - \int_0^x C_0 dx; \quad C_2 \frac{x^2}{2!} = - \int_0^x C_1 x dx.$$

$$C_{r+1} \frac{x^{r+1}}{(r+1)!} = - \int_0^x C_r \frac{x^r}{r!} dx = (-)^{r+1} \int_0^x C_0 dx.$$

Hence the series* may be written

$$\varphi x = C_0 + \sum_{i=0}^{i=n} \int_0^x C_i dx + \frac{x^{n+1}}{(n+1)!} \varphi^{n+1} u. \quad (10)$$

* Since the only condition in general imposed on the quantities a_1, \dots, a_n , is that they shall be functions independent of x , a number of series may be deduced from the general one through suppositions regarding these arbitraries. While I have only noticed the simple particular cases of Taylor's, Maclaurin's, and Bernoulli's formulæ, I have prepared for publication a paper dealing with this series alone in which other forms are noticed, the consideration of which would be beyond the design of the present paper.

In the determinant (4) let φx be a rational integral function of the $(n+1)$ th degree. And let

$$\varphi a_0 = 0, \quad \varphi' a_1 = 0, \quad \dots, \quad \varphi^n a_n = 0;$$

so that α_0 is a root of $\varphi x = 0$, and $\alpha_1, \dots, \alpha_n$ are roots of its first n derived functions respectively.

Then

$$\varphi^{n+1}u = (n+1)!;$$

and we have the identity

$$\varphi x \equiv (-1)^{n+1} (n+1)! \left| \begin{array}{l} 1, x, \dots, \frac{1}{(n+1)!} x^{n+1} \\ 1, a_0, \dots, \frac{1}{(n+1)!} a_0^{n+1} \\ 0, 1, \dots, \frac{1}{n!} a_1^n \\ . \quad . \quad . \quad . \quad . \quad . \\ 0, 0, \dots, 1, a_n \end{array} \right|$$

If, therefore, we know a root of any equation and a root of each of its derivatives, we may at once write down the function from the above.

Also, since we can always factor out at sight $x - \alpha_0$ from the right hand member of the identity, we can write down immediately the first depressed equation of $\varphi x = 0$, which has for its roots the other n roots of $\varphi x = 0$, without actually performing division.

If $x = 0$, then

$$\varphi(0) = (-1)^{n+1} (n+1)! \begin{vmatrix} \alpha_0, & \frac{1}{2!} \alpha_0^2, & \dots, & \frac{1}{(n+1)!} \alpha_0^{n+1} \\ 1, & \alpha_1, & \dots, & \frac{1}{n!} \alpha_1^n \\ 0, & 1, & \dots, & \frac{1}{(n-1)!} \alpha_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \dots, & 1, \quad \alpha_n \end{vmatrix}$$

is a homogeneous relation between the roots, and the member on the right is the product of all the roots of $\varphi x = 0$, with their signs changed.

The most general power series of a function in terms of its variable, which can be conceived is

$$fx = ax^a + bx^b + cx^c + \dots + zx^z,$$

where $\alpha, \beta, \gamma, \dots$ are positive integers, and a, b, c, \dots are functions independent of x .

If the series be convergent, it may extend to an infinite number of terms ; otherwise the number of terms must be finite, and instead of a series we have a formula of relation.

Letting D_u^{n+1} denote the result of differentiating $n + 1$ times with respect to x the determinant D and in the result replacing x by u , we have, with like meaning for \mathcal{A}_u^{n+1} ,

$$D_u^{n+1} = \frac{\mathcal{A}_u^{n+1}}{\mathcal{A}_0} D_0,$$

or

$$D_0 = \frac{D_u^{n+1}}{\mathcal{A}_u^{n+1}} \mathcal{A}_0.$$

Drop the suffix because x_0 is any value of x and put

$$\frac{D_u^{n+1}}{\mathcal{A}_u^{n+1}} = (-1)^{n+1} Fu.$$

We have finally

$$\left| \begin{array}{cccccc} fx, & \frac{x^a}{a!}, & \frac{x^\beta}{\beta!}, & \frac{x^\gamma}{\gamma!}, & \cdots, & \frac{x^\zeta}{\zeta!}, & \frac{x^{\zeta+\eta}}{(\zeta+\eta)!} \\ f\omega_0, & \frac{\omega_0^a}{a!}, & \frac{\omega_0^\beta}{\beta!}, & \frac{\omega_0^\gamma}{\gamma!}, & \cdots, & \frac{\omega_0^\zeta}{\zeta!}, & \frac{\omega_0^{\zeta+\eta}}{(\zeta+\eta)!} \\ f'\omega_1, & \frac{\omega_1^{a-1}}{(a-1)!}, & \frac{\omega_1^{\beta-1}}{(\beta-1)!}, & \frac{\omega_1^{\gamma-1}}{(\gamma-1)!}, & \cdots, & \frac{\omega_1^{\zeta-1}}{(\zeta-1)!}, & \frac{\omega_1^{\zeta+\eta-1}}{(\zeta+\eta-1)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f^n \omega_n, & \frac{\omega_n^{a-n}}{(a-n)!}, & \frac{\omega_n^{\beta-n}}{(\beta-n)!}, & \frac{\omega_n^{\gamma-n}}{(\gamma-n)!}, & \cdots, & \frac{\omega_n^{\zeta-n}}{(\zeta-n)!}, & \frac{\omega_n^{\zeta+\eta-n}}{(\zeta+\eta-n)!} \\ Fu, & 0, & 0, & 0, & \cdots, & 0, & 1 \end{array} \right| \equiv 0.$$

This is the general expression sought, and determines the forms which the coefficients a, b, c, \dots must have.

More generally still, being given any $n + 3$ functions of x , such as

$$a, \beta, \gamma, \dots, \theta, \dots, \zeta, \varphi,$$

form a determinant F as before with $n + 2$ of these functions omitting the last φ . Then form a new determinant exactly as before in which however we replace the r th function θ by the function φ , which was previously omitted, and call this latter determinant \mathcal{A} .

Put

$$F = \mathcal{A}R,$$

R being an unknown function of x .

If $x = x_0$,

$$F_0 = \mathcal{A}_0 R_0$$

is independent of x .

The $(n + 1)$ th derivative of the function

$$F - \Delta \frac{R_0}{\Delta_0},$$

vanishes for some value of x , say u , which lies in magnitude between the greatest and least of the values a, b, c, \dots, z . So that

$$F_u^{n+1} = \Delta_u^{n+1} \frac{R_0}{\Delta_0}.$$

Put

$$\frac{F_u^{n+1}}{\Delta_u^{n+1}} = \psi_u^{n+1}.$$

Then, dropping the suffix, we have

$$F = \psi_u^{n+1} \Delta.$$

Transposing the r th column to the first in F , and to the last in Δ we have the relation*

$$\begin{vmatrix} \theta, & a, & \beta, & \dots, & \zeta, & \varphi \\ \theta_a, & a_a, & \beta_a, & \dots, & \zeta_a, & \varphi_a \\ \theta'_b, & a'_b, & \beta'_b, & \dots, & \zeta'_b, & \varphi'_b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \theta_z^n, & a_z^n, & \beta_z^n, & \dots, & \zeta_z^n, & \varphi_z^n \\ \psi_u^{n+1}, & 0, & 0, & \dots, & 0, & 1 \end{vmatrix} \equiv 0,$$

if the functions a, β, γ, \dots and their derivatives are finite and continuous.

The great comprehensiveness of this formula is such that it would lead us altogether beyond the limits of this paper† to do more than consider one form of a particular instance of the simplest possible case; namely, that of three functions θ, a , and β .

* While I have written out the first row in the variable as functions of x , and the second row as functions in which x is replaced by a , a quantity which is independent of x , and then proceeded with the derivatives, we might have inserted previous to differentiation a number of function rows in which x is replaced by quantities v, y, z etc, independent of x , thus making the determinant even more markedly an *alternant-differentiate*. This determinant I venture to name a "*composite*," inasmuch as it is composed partially of the form of an alternant and partially of a form which is that of a Wronskian. With this distinction, however, in the case of the latter; that the quantities under the derived functional signs may in general be entirely independent of each other and are independent of the corresponding quantity, x , under the primitive functional sign.

† I have prepared a paper upon the application of the *composite* to the expansion of an arbitrary function in terms of certain transcendental functions, forms of $\sin x$ and $\cos x$, which presents many interesting points.

Here we have, changing the notation to more familiar forms, the relation

$$\frac{\varphi x f y - f x \varphi y}{f x \phi y - \phi x f y} = \frac{\varphi' u f y - f' u \varphi y}{f' u \phi y - \phi' u f y}, \quad (\text{i})$$

where u lies between x and y .

In particular, if the function ϕ is a constant, say c , then we have

$$\frac{\varphi x f y - f x \varphi y}{f x - f y} = \frac{\varphi' u f y - f' u \varphi y}{f' u}. \quad (\text{ii})$$

We know that

$$f x - f y = (x - y) f' v,$$

where v lies between x and y ; therefore

$$\frac{\varphi x f y - f x \varphi y}{x - y} = \frac{f' v}{f' u} (\varphi' u f y - f' u \varphi y). \quad (\text{iii})$$

The member on the left is the form of Professor Cayley's resultant.

If φ is also a constant, say, b , then (iii) reduces to the familiar form of Lagrange's formula,

$$f x - f y = (x - y) f' v.$$